assume that $g_l(t) \leqslant G_l$, $h_l(t) \leqslant H_l$), and

$$\sum_{l=1}^{\infty} G_l < 1, \quad \sum_{l=1}^{\infty} H_l < \infty, \quad \sum_{l=1}^{\infty} \int_{t_0}^{\infty} g_l(\tau)\, d\tau < \infty, \quad \sum_{l=1}^{\infty} \int_{t_0}^{\infty} h_l(\tau)\, d\tau < \infty$$

Theorems 1 – 3 will hold for this case if conditions ω are replaced by conditions ω_1.

REFERENCES

1. Krasovskii, N. N., Some Problems in the Theory of Stability of Motion. Moscow, Fizmatgiz, 1959. (See also English translation, Krasovskii, N. N., Stability of Motion, Stanford University Press, Stanford, California, 1963).
2. Bellman, R., Stability Theory of Differential Equations. New York, McGraw-Hill Book Co., Inc., 1953.
3. El'sgol'ts, L. E. and Norkin, S. B., Introduction to the Theory of Differential Equations with a Deviating Argument. Moscow, "Nauka", 1971.

Translated by N. H. C.

UDC 531.36

ON STABILIZATION OF POTENTIAL SYSTEMS

PMM Vol. 39, №1, 1975, pp. 53-58
V. M. LAKHADANOV
(Minsk)
(Received April 11, 1974)

We prove a generalization of the Kelvin-Chetaev theorem. We examine certain aspects of the stabilization of unstable potential systems by gyroscopic and nonconservative forces [1].

1. We consider the systems (D, F are constant symmetric ($n \times n$)-matrices)

$$x^{\cdot\cdot} + Dx^{\cdot} + Fx = 0 \tag{1.1}$$

$$x^{\cdot\cdot} + Dx^{\cdot} + Fx = X(x, x^{\cdot}) \tag{1.2}$$

$$x = \operatorname{col}(x_1, \ldots, x_n),\ X(x, x^{\cdot}) = \operatorname{col}(X_1(x, x^{\cdot}), \ldots, X_n(x, x^{\cdot})),\ X(0,0) \equiv 0$$

(the functions $X_i(x, x^{\cdot})$ contain x_i, $x_i^{\cdot}$ to powers not less than second).

A result which we can state as the following theorem was proved in [2].

The Kelvin-Chetaev theorem. If matrix D is positive definite and among the eigenvalues of matrix F there is at least one negative, then systems (1.1) and (1.2) are unstable.

A result which can be looked upon as a generalization of the Kelvin-Chetaev theorem was proved in [3].

The theorem from [3]. If matrix D is positive definite and $|F| \neq 0$, then the number of roots with a positive real part of the characteristic equation

$$| E\lambda^2 + D\lambda + F | = 0 \tag{1.3}$$

equals the number of negative eigenvalues of matrix F.

Using the substitution $x = ye^{\varepsilon t}$ suggested in [2], where ε is a suitably chosen constant, we can get rid of the condition $|F| \neq 0$ and prove that with a positive-definite matrix D the number of roots of characteristic equation (1.3) with a positive real part is not less than the number of negative eigenvalues of matrix F. But it turns out that there exists a simple proof of the following stronger statement.

Theorem 1.1. The number of positive roots of characterisitic equation (1.3) is not less than the number of negative eigenvalues of matrix F.

Proof. We consider the symmetric matrix

$$B(\lambda) = E\lambda^2 + D\lambda + F$$

where λ is a real parameter. The determinant of this matrix equals the product of its eigenvalues

$$|B(\lambda)| = b_1(\lambda)\cdot b_2(\lambda)\ \ldots\ b_n(\lambda)$$

For a sufficiently large $\lambda > 0$ the matrix $B(\lambda)$ is positive definite and its eigenvalues are greater than zero. If among the eigenvalues F $b_1(0), b_2(0), \ldots, b_n(0)$ of matrix F there are negative ones, for example, $b_k(0) < 0$, then by virtue of their continuity with respect to λ there exists $\lambda_0 > 0$ such that $b_k(\lambda_0) = 0$, i.e. characteristic equation (1.3) has a positive root λ_0. The completion of the proof is obvious.

Note. It can be proved analogously that the number of negative roots of characteristic equation (1.3) is not less than the number of negative eigenvalues of matrix F.

Corollaries. 1. If matrix F is negative definite, then the characterisitc equation (1.3) has n positive roots and n negative roots.

2. If among the eigenvalues of matrix F there is at least one negative, then systems (1.1) and (1.2) are unstable.

2. Let us consider the system

$$x^{\cdot\cdot} + HGx^{\cdot} + Fx = 0 \tag{2.1}$$

where G is a constant skew-symmetric ($n \times n$)-matrix characterizing gyroscopic forces, H is a real parameter. Systems of form (2.1) have been exhaustively treated in [4].

Theorem 2.1. If matrix F is negative definite and the determinant $|G| \neq 0$, then system (2.1) is stable for $H > H_0 > 0$, where H_0 is defined by the equality

$$H_0 = \frac{f_1g_1 - 1}{2f_1g_2} + \left[\left(\frac{f_1g_1 - 1}{2f_1g_2}\right)^2 + \frac{f_1 - f_2}{g_2}\right]^{1/2} \tag{2.2}$$

in which f_1 and g_1 are the largest eigenvalues of matrices F and $G'FFG$, f_2 and g_2 are the smallest eigenvalues of matrices F and $G'G$.

Proof. System (2.1) has the energy integral

$$V_1 = {}^1/_2\,(x^{\cdot\prime}Ex^{\cdot} + x'Fx)$$

It is easily verified that it has a second quadratic integral

$$V_2 = (HGx^{\cdot} + Fx)'\ (HGx^{\cdot} + Fx) + x^{\cdot\prime}Fx^{\cdot}$$

We set up the linear sheaf of these integrals

$$V = 2\,(H/f_1 - f_1)\ V_1 + V_2$$

which can be written in the form

$$V = x^{\cdot\prime} [H^2G'G - HG'FFG + (H/f_1 - f_1) E + F] x^{\cdot} +$$
$$x' [(H/f_1 - f_1) F - HE + FF] x + H (x + FGx^{\cdot})' (x + FGx^{\cdot})$$

It can be shown that the quadratic form V is positive definite for $H > H_0$. Then, system (2.1) is stable by virtue of Liapunov stability theorem.

If matrix F is negative definite and all its eigenvalues are equal to each other, then system (2.1) is unstable when $|G| = 0$. However, in the general case stable systems (2.1) exist in which matrix F is negative definite and $|G| = 0$. The following example proves this. Let us consider system (2.1) in which

$$H = 1, \quad G = \begin{Vmatrix} 0 & 1 & 10 & 0 \\ -1 & 0 & 0 & 10 \\ -10 & 0 & 0 & 100 \\ 0 & -10 & -100 & 0 \end{Vmatrix}, \quad -F = \begin{Vmatrix} 10^{-6} & 0 & 0 & 0 \\ 0 & 10^{-6} & 0 & 0 \\ 0 & 0 & 10^3 & 0 \\ 0 & 0 & 0 & 10^3 \end{Vmatrix}$$

Here matrix F is negative definite, $|G| = 0$. The characteristic equation of this system has distinct pure imaginary roots, i.e. the system is stable. Note that in this example the system is unstable for $H = 0$, becomes stable as H increases $(H = 1)$, and becomes and remains unstable under a subsequent increase in H.

3. We consider the systems (P is a constant skew-symmetric ($n \times n$)-matrix characterizing nonconservative forces)

$$x^{\cdot\cdot} + Gx^{\cdot} + Fx + Px = 0. \tag{3.1}$$

$$x^{\cdot\cdot} + Gx^{\cdot} + Fx + Px = X(x, x^{\cdot}) \tag{3.2}$$

About systems (3.1) and (3.2) we know:

1) system (3.1) is not asymptotically stable [4];

2) if matrix F is negative definite, then systems (3.1) and (3.2) are unstable for odd n [4];

3) if matrix F is negative definite and $P = \alpha G$ $(\alpha > 0)$, then systems (3.1) and (3.2) are unstable [5].

Note. The restriction $\alpha > 0$ is unessential and statement (3) is valid for any constant $\alpha \neq 0$.

Theorem 3.1. If $F = kE$, where k is an arbitrary constant, and matrices G and P are commutative, then system (3.1) is unstable.

Theorem 3.2. If the hypotheses of Theorem 3.1 are fulfilled and $|P| \neq 0$, then system (3.2) is also unstable.

The proofs of Theorems 3.1 and 3.2 agree with the proofs of Theorems 4 and 5 of [6].

Theorem 3.3. Systems (3.1) and (3.2) are unstable if

$$\sum_{i,j} g_{ij} p_{ij} \neq 0 \tag{3.3}$$

where g_{ij}, p_{ij} are the elements of matrices G, P.

Proof. Consider the characteristic equation

$$| E\lambda^2 + G\lambda + F + P | = \lambda^{2n} + a_1\lambda^{2n-1} + \ldots + a_{2n} = 0$$

It can be shown that

$$a_1 = 0, \qquad a_3 = \sum_{i,j} g_{ij} p_{ij}$$

If condition (3.3) is fulfilled, then $a_3 \neq 0$, which implies that there is at least one root with a nonzero real part in the characteristic equation. But then from the condition $a_1 = 0$ it follows that the characteristic equation has at least one root with a positive real part. The theorem is proved.

Corollaries. 1. If $G = \alpha P$, then systems (3.1) and (3.2) are unstable.

2. Systems (3.1) and (3.2) are unstable for $n = 2$.

Examples. 1. Consider system (3.1) in which

$$G = \begin{Vmatrix} 0 & -6 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{Vmatrix}, \quad -F = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{Vmatrix}, \quad P = \begin{Vmatrix} 0 & 0 & 0 & -\alpha \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{Vmatrix}$$

where $\alpha = 0.01$. We can show that the characteristic equation of this system has distinct pure imaginary roots. Thus, this system is stable. Here matrix F is negative definite, matrices G and P are commutative, $|P| \neq 0$, but Theorems 3.1 and 3.2 are inoperative since $F \neq kE$.

2. As already noted, it is impossible to stabilize the system

$$x^{\cdot\cdot} + Fx = 0 \tag{3.4}$$

with $F = \alpha E$ $(\alpha < 0)$ by gyroscopic forces with determinant $|G| = 0$. In this case it is impossible to stabilize it with only nonconservative forces. However, the simultaneous use of gyroscopic and nonconservative forces enables us to achieve the stabilization even under the condition $|G| = 0$. In fact, consider system (3.1) in which

$$F = \alpha E, \quad G = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}, \quad P = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{Vmatrix}$$

where $\alpha = -0.01$. We can show that the characterisitc equation of this system has distinct pure imaginary roots, i.e. the system is stable

4. The problem of stabilizing unstable potential systems (3.4) by nonconservative forces was posed in [6]. It was also given the solution for cases $n = 2$ and 3. Below we solve this problem in the general case.

Theorem 4.1. A real matrix R can be reduced by an orthogonal transformation to a form with distinct positive diagonal elements if and only if

$$\operatorname{sp} R > 0 \tag{4.1}$$

$$R + R' \neq \alpha E \tag{4.2}$$

Proof. Since the trace of a matrix is an invariant of an orthogonal transformation, the necessity of condition (4.1) is obvious. We write matrix R as a sum of its symmetric and skew-symmetric parts

$$R = {}^1/_2 (R + R') + {}^1/_2 (R - R') \tag{4.3}$$

Symmetry and skew-symmetry are preserved under an orthogonal transformation; therefore, from (4.3) follow the necessity of condition (4.2) and the fact that without loss of generality we can assume matrix R as a diagonal one. Let $R = \operatorname{diag}(r_1, \ldots, r_n)$. We prove the sufficiency of conditions (4.1) and (4.2) by induction in n.

For $n = 2$ it suffices to consider the case $r_1 > 0$, $r_2 \leqslant 0$. To matrix R we apply an orthogonal transformation with matrix $A : ARA' = S$. We can show that the formulas

$$a_{11}^2 = a_2^{22} = \frac{\varepsilon - r_2}{r_1 - r_2}, \qquad a_{12}^2 = a_{21}^2 = \frac{r_1 - \varepsilon}{r_1 - r_2}$$

with a sufficiently small $\varepsilon > 0$ determine the elements of the matrix A of the orthogonal transformation reducing matrix R to the required form.

Suppose that the theorem has been proved for all $n \leqslant m$. Let us also prove its validity for $n = m + 1$. Without loss of generality we assume that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_{m+1}$.

Case $r_{m+1} > 0$.

1) If $r_2 \neq r_{m+1}$, then using an orthogonal transformation with matrix

$$A = \begin{Vmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & A^m \\ 0 & \end{Vmatrix} \tag{4.4}$$

where A^m is an orthogonal ($m \times m$)-matrix, the matrix R can be reduced by the induction assumption to the form

$$Q = ARA' = \begin{Vmatrix} q_{11} \cdots q_{1n} \\ \vdots \quad Q^m \\ q_{n1} \end{Vmatrix} \tag{4.5}$$

where $q_{11} > 0$ and the ($m \times m$)-matrix Q^m has distinct positive diagonal elements.

2) If $r_2 = r_{m+1}$, then we make r_1 and r_2 change places and reduce the matrix obtained to form (4.5) by the same method.

Case $r_{m+1} \leqslant 0$. We make r_2 and r_{m+1} change places anf to the matrix R_1 obtained we apply an orthogonal transformation with matrix

$$A_1 = \begin{Vmatrix} a_{11} & a_{12} & 0 \ldots 0 \\ a_{21} & a_{22} & 0 \ldots 0 \\ 0 & 0 & \\ \vdots & \vdots & E \\ 0 & 0 & \end{Vmatrix} \tag{4.6}$$

Analogously to the case $n = 2$ we reduce matrix R_1 to the form

$$R_2 = A_1 R_1 A_1' = \begin{Vmatrix} \varepsilon \ldots \\ \vdots \; R_3 \end{Vmatrix}$$

where $\varepsilon > 0$ and the ($m \times m$)-matrix R_3 satisfies conditions (4.1) and (4.2). Matrix R_2 can by the inductive assumption be reduced to the form (4.5) by an orthogonal transformation with a matrix of the form (4.4).

Thus, under the assumptions made, the matrix R can always be reduced to the form

(4.5) by an orthogonal transformation. If the q_{11} in (4.5) does not agree with any of the remaining distinct diagonal elements, then the theorem is proved. However, if q_{11} agrees with one of them, then without loss of generality, we can assume that $q_{11} \neq q_{22}$. In this case we can show that an orthogonal transformation with a matrix of the form (4.6) exists, which reduces matrix Q to the required form. The theorem is completely proved.

Theorem 4.2. The unstable system (3.4) can be stabilized by nonconservative forces if and only if the condition

$$\operatorname{sp} F > 0 \tag{4.7}$$

is satisfied.

Proof. The necessity of condition (4.7) was proved in [6]. If system (3.4) is unstable and condition (4.7) is satisfied, the matrix F satisfies the hypotheses of Theorem 4.1. Therefore, without loss of generality we can assume that matrix F is already reduced to a form with distinct positive diagonal elements. Then we choose the stabilizing matrix P in the system

$$x^{\cdot\cdot} + Fx + Px = 0 \tag{4.8}$$

in the following manner. For $i < j$ we set $p_{ij} = f_{ij}$ and for $i > j$ we set $p_{ij} = -f_{ij}$. For such a choice of the skew-symmetric matrix P the roots of the characteristic equation of system (4.8) are pure imaginary and distinct, i.e. system (4.8) is stable. The theorem is proved.

Problem 4.1. Determine the conditions under which the unstable system

$$x^{\cdot\cdot} + Dx^{\cdot} = 0 \tag{4.9}$$

can be stabilized by gyroscopic forces, i.e. select a constant skew-symmetric matrix G so that the system

$$x^{\cdot\cdot} + Dx^{\cdot} + Gx^{\cdot} = 0 \tag{4.10}$$

is stable.

In [4] it was shown that gyroscopic stabilization of unstable systems (4.9) is impossible under the condition $\operatorname{sp} D < 0$.

Theorem 4.3. If $\operatorname{sp} D > 0$, an unstable system (4.9) can be stabilized by gyroscopic forces.

This theorem is proved similarly to Theorem 4.2. We note that the condition $\operatorname{sp} D > 0$ is not necessary for the gyroscopic stabilization of unstable systems (4.9) and even for $n = 2$ stable systems (4.10) exist in which $\operatorname{sp} D = 0$.

The author thanks V.V.Rumiantsev for advice and remarks on this paper.

REFERENCES

1. Merkin, D.R., Introduction to the Theory of Stability of Motion. Moscow, "Nauka," 1971.
2. Chetaev, N.G., The Stability of Motion. (English translation), Pergamon Press, Book № 09505, 1961.
3. Zajac, E.E., The Kelvin-Tait-Chetaev theorem and extensions. J. Astronaut. Sci., Vol. 11, №2, 1964.
4. Merkin, D.R., Gyroscopic Systems. Moscow, Gostekhizdat, 1956.
5. Agafonov, S.A., On the stability of nonconservative systems. Vestn. Moskovsk. Gos. Univ., Ser. Matem., Mekhan., №4, 1972.
6. Lakhadanov, V.M., On the influence of structure of forces on the stability of motion. PMM Vol. 38, №2, 1974.

Translated by N.H.C.